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Static perfect fluids in general relativity

A BARNES†

Department of Mathematics, Imperial College of Science and Technology, London SW7, UK

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Abstract. All solutions of Einstein's field equations which represent a static perfect fluid are considered and a number of results for vacuum fields obtained by Ehlers and Kundt are generalized. In §§ 5 to 7 all such solutions with a degenerate Weyl tensor are found explicitly. As is the case with vacuum fields of Ehlers and Kundt all the solutions obtained are found to admit at least a two-parameter group of local isometries.

1. Introduction

In this paper I shall consider static solutions of Einstein's equations

$$R_{\alpha\beta}-\tfrac{1}{2}Rg_{\alpha\beta} = -T_{\alpha\beta} \tag{1}$$

Greek indices running from 1 to 4, where $T_{\alpha\beta}$ is the energy–momentum tensor of a perfect fluid. Consequently

$$T_{\alpha\beta} = (p+\rho)u_\alpha u_\beta - pg_{\alpha\beta} \tag{2}$$

where u_α, ρ and p are respectively the four velocity, energy density and the pressure of the fluid. If there exists a nonzero cosmological constant Λ it may be absorbed in the tensor $T_{\alpha\beta}$ as follows:

$$p = p_{\text{matter}}+\Lambda \qquad \rho = \rho_{\text{matter}}-\Lambda. \tag{3}$$

All exact solutions of the equations (1) and (2) which have a degenerate Weyl tensor will be obtained by a method that generalizes that used by Levi-Civita (1917–9) and Ehlers and Kundt (1962, to be referred to as EK) to analyse vacuum fields.

2. Theorems on static space–times

We shall appeal to several general theorems on static space–times which we list below. The proofs of these theorems are to be found in EK.

A space–time is called static if it admits a time-like Killing vector ξ_α which is hypersurface orthogonal, that is

$$\xi_\alpha\xi^\alpha = -e^{2U} < 0 \qquad \xi_{(\alpha;\beta)} = 0 \qquad \xi_{[\alpha;\beta}\xi_{\gamma]} = 0. \tag{4}$$

If u^α is the unit vector parallel to ξ^α it can be shown that

$$u_\alpha u^\alpha = -1 \qquad u_{\alpha;\beta} = -\dot{u}_\alpha u_\beta \qquad \dot{u}_{[\alpha}u_{\beta]} = 0 \tag{5}$$

† Now at School of Mathematics, Merz Court, The University, Newcastle-upon-Tyne, UK.

where a dot signifies covariant differentiation along u^α. Equation (5) means that the integral curves of u^α form a normal rigid congruence along which $\dot{u}_\alpha$ is Fermi-propagated (Synge 1960).

Space–times may be classified (Petrov 1962) by considering certain canonical forms of the Weyl tensor defined by

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\alpha\beta\epsilon[\gamma}R^{\epsilon}_{\delta]} + \tfrac{1}{6}Rg_{\alpha\beta\gamma\delta}.$$

Tetrad components are taken with respect to an orthonormal tetrad of vectors (e^α_A, e^α_4), where e^α_4 is a time-like vector and capital Latin indices run from 1 to 3. We consider the canonical forms of the complex tensor D_{AB} defined by

$$D_{AB} = E_{AB} + \mathrm{i}H_{AB} = -(C_{A4B4} + \mathrm{i}^*C_{A4B4}).$$

The Weyl tensor is said to be of Petrov type I, II or III, respectively, if D_{AB} when regarded as a linear transformation on a three dimensional complex vector space has invariant subspaces of dimension 1, 2 or 3 respectively.

It may be shown (EK) from equation (5) that the vector u^α is an eigenvector of $R_{\alpha\beta}$ with eigenvalue $\dot{u}^\gamma_{;\gamma}$. If we choose $e^\alpha_4 = u^\alpha$, since $H_{\alpha\gamma} = -{}^*C_{\alpha\beta\gamma\delta}u^\beta u^\delta = 0$, it follows that u^α is a principal vector of the Weyl tensor and, furthermore, that the Weyl tensor is of type I with vanishing pseudoscalars. This follows since D_{AB} is real and may consequently be diagonalized by a rotation of the space-like triad (e^α_A).

If the Weyl tensor is nondegenerate, the principal tetrad is uniquely determined up to reflections and so u^α is unique. In the degenerate case, the vector u^α, being an eigenvector of $R_{\alpha\beta}$ and consequently of $T_{\alpha\beta}$, is unique if $p+\rho \neq 0$. If, however, $p+\rho = 0$, the vector u^α need not be unique.

In EK it is shown that coordinates may be chosen in such a way that the metric G takes the form

$$G = \tilde{g}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b - \exp(2U(x^a))\,\mathrm{d}t^2 \tag{6}$$

where lower case Latin indices run from 1 to 3 and $t = x^4$. In this coordinate system $u^\alpha = \mathrm{e}^{-U}\delta^\alpha_4$ and the coordinates are unique (if u^α is unique) up to the transformations

$$\tilde{x}^a = \tilde{x}^a(x^b) \qquad \tilde{t} = at+b \tag{7}$$

where a and b are constants.

The field equations (1) become

$$\tilde{R}_{ab} + \mathrm{e}^{-U}(\mathrm{e}^U)_{\|ab} = \tfrac{1}{2}(p-\rho)\tilde{g}_{ab} \tag{8}$$

$$\mathrm{e}^{-U}(\mathrm{e}^U)^{\|a}_{\|a} = \tfrac{1}{2}(\rho+3p) \tag{9}$$

where $\tilde{R}_{ab}$ is the Ricci tensor of $\tilde{g}_{ab}$ and $\|$ signifies a covariant derivative with respect to the metric $\tilde{g}_{ab}$.

It follows from equations (8) and (9) that

$$R = -2\rho. \tag{10}$$

The contracted Bianchi identities (the equations of hydrostatic equilibrium) are

$$p_{|a} = -U_{|a}(p+\rho) \tag{11}$$

where $|$ signifies ordinary differentiation. It follows from equation (11) that

$$p_{[|a}U_{|b]} = \rho_{[|a}U_{|b]} = 0 \tag{12}$$

and so if U is not constant both p and ρ are functions of U only. A second consequence of equation (11) is that both p and ρ are constant when $p+\rho = 0$. From equation (3), we can interpret solutions with $p+\rho = 0$ as vacuum solutions with a cosmological constant.

It will be convenient to express the metric (6) in the form

$$G = e^{-2U}(g'_{ab}\,dx^a\,dx^b) - e^{2U}\,dt^2. \tag{6$'$}$$

Using equation (A.1) in the Appendix with $n = 3$, $V = e^{-U}$, the field equations (8) and (9) become

$$R'_{ab} + 2U_{|a}U_{|b} = 2p\,e^{-2U}g'_{ab} \tag{8$'$}$$

and

$$U|^{a}_{\|a} = \tfrac{1}{2}(\rho+3p)\,e^{-2U} \tag{9$'$}$$

where R'_{ab} is the Ricci tensor of g'_{ab} and covariant derivatives are taken with respect to g'_{ab}.

Taking tetrad components with respect to a Weyl principal tetrad $(e^{\alpha}_{A}, e^{\alpha}_{4} = u^{\alpha})$ and remembering that the Weyl tensor is of type I with $H_{AB} = 0$, we obtain

$$g_{AC} = \delta_{AC} \qquad g_{44} = -1 \qquad g_{A4} = 0$$

and (13)

$$E_{AC} = -C_{A4C4} = \sum_{M=1}^{3} \alpha_M \delta_{MA}\delta_{MC} \qquad \sum \alpha_M = 0$$

where the α_M are the Weyl principal scalars. Since

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + g_{\alpha\beta\epsilon[\gamma}R^{\epsilon}_{\delta]} + \tfrac{1}{6}Rg_{\alpha\beta\gamma\delta}$$

we have

$$E_{AC} = -(R_{A4C4} + \tfrac{1}{2}g_{AC}R^4_4 + \tfrac{1}{2}R_{AC} - \tfrac{1}{6}Rg_{AC}).$$

Using the equation $R_{A4C4} = \tilde{R}_{AC} - R_{AC}$, we obtain

$$E_{AC} = \tfrac{1}{2}R_{AC} - \tilde{R}_{AC} - \tfrac{1}{2}\delta_{AC}(R^4_4 - \tfrac{1}{3}R).$$

With the aid of equations (1), (2) and (13), we obtain

$$\tilde{R}_{AC} = -\tfrac{2}{3}\rho\delta_{AC} - \sum_{M}\alpha_M\delta_{MA}\delta_{MC} \tag{14}$$

or equivalently

$$P_{AC} = -\sum_{M}\alpha_M\delta_{MA}\delta_{MC}$$

where P_{AC} is the trace-free part of $\tilde{R}_{AC}$. Thus we see that a space-like triad of Weyl principal vectors form an eigen-triad of the Ricci tensor of the 3-spaces (t = constant) with eigenvalues $-\tfrac{2}{3}\rho - \alpha_M$.

3. Integrability conditions

In § 4, in order to solve the field equations (8) and (9) we shall need their integrability conditions.

Using equation (8) we obtain

$$\tilde{R}_{a[b\|c]} = \mathrm{e}^{-U}(\mathrm{e}^{U})_{\|a[b}U_{|c]} - \mathrm{e}^{-U}(\mathrm{e}^{U})_{\|a[bc]} + \tfrac{1}{2}(p-\rho)_{|[c}\tilde{g}_{b]a}.$$

Eliminating second and third derivatives of U by means of equations (8), (10) and the Ricci identity, we obtain

$$L_{a[b\|c]} = -R_{a[b}U_{|c]} + \tfrac{1}{2}(p-\rho)\tilde{g}_{a[b}U_{|c]} + \tfrac{1}{2}g_{a[b}p_{|c]} + \tfrac{1}{2}U^{|d}\tilde{R}_{dabc}$$

where

$$L_{ab} = \tilde{R}_{ab} - \tfrac{1}{4}\tilde{R}\tilde{g}_{ab}.$$

Using equation (11) and the identity valid for spaces of dimension 3, namely $\tilde{R}_{abcd} = -\eta_{abe}(\tilde{R}^{ef} - \tfrac{1}{2}\tilde{R}g^{ef})\eta_{fcd}$, we obtain

$$L^{a}_{[b\|c]} = -U_{|d}(\delta^{a}_{[b}P^{d}_{c]} + 2\delta^{d}_{[c}P^{a}_{b]}). \tag{15}$$

On taking triad components of equation (15) and suspending the summation convention, it follows from equation (14) that

$$2L_{A[B.C]} = -(\alpha_B - \alpha_A)\gamma_{BAC} + (\alpha_C - \alpha_A)\gamma_{CAB} + \alpha_{C.B}\delta_{AC} - \alpha_{B.C}\delta_{AB} - \tfrac{1}{3}\delta_{A[B}\rho_{.C]}$$

and

$$2L_{A[B.C]} = \delta_{AB}U_{.C}(\alpha_C + 2\alpha_B) - \delta_{AC}U_{.B}(\alpha_B + 2\alpha_C)$$

where $\gamma_{ABC} = e_{A\mu;\nu}e^{\mu}_{B}e^{\nu}_{C}$ are the Ricci rotation coefficients of the triad (e^{μ}_{A}) and where a dot signifies the triad component of a covariant derivative.

The equations above are equivalent to

$$\gamma_{123}(\alpha_1 - \alpha_2) = \gamma_{231}(\alpha_2 - \alpha_3) = \gamma_{312}(\alpha_3 - \alpha_1) = E \tag{16}$$

say, and

$$\alpha_{A.C} + (\alpha_A - \alpha_C)\gamma_{CAA} + U_{.C}(\alpha_A - \alpha_B) + \tfrac{1}{6}\rho_{.C} = 0 \qquad \text{for } A, B, C \neq . \tag{17}$$

Using equations (13), (17), we obtain

$$\alpha_{A.A} + \sum_{D}(\alpha_A - \alpha_D)\gamma_{ADD} - \tfrac{1}{3}\rho_{.A} = 0. \tag{18}$$

The classification of vacuum, nondegenerate fields (Jordan *et al* 1960, to be referred to as JEK) can be extended immediately to nondegenerate perfect fluid fields since equation (16) is identical with equation (3.1.16) of JEK.

If $E = 0$, then $\gamma_{123} = \gamma_{321} = \gamma_{312} = 0$, and so the Weyl tetrad is hypersurface orthogonal. If, however, $E \neq 0$, only $e^{\alpha}_{4} = u^{\alpha}$ of the tetrad is hypersurface orthogonal.

For type D fields ($\alpha_1 = \alpha_2 = -\tfrac{1}{2}\alpha$) we see from equations (16) and (17) that $\gamma_{231} = \gamma_{312} = 0$ and $\gamma_{311} = \gamma_{322}$. Hence we can use the result obtained in JEK that, by a suitable rotation about e_3, e_1 and e_2 can be made hypersurface orthogonal. Hence, a coordinate system can be chosen so that

$$G = (\mathrm{e}^{\beta_1}\,\mathrm{d}x^1)^2 + (\mathrm{e}^{\beta_2}\,\mathrm{d}x^2)^2 + (\mathrm{e}^{\beta_3}\,\mathrm{d}x^3)^2 - (\mathrm{e}^{U}\,\mathrm{d}t)^2. \tag{19}$$

In this coordinate system

$$e^{\mu}_{A} = \mathrm{e}^{-\beta_A}\delta^{\mu}_{A} \qquad \gamma_{ABC} = 0 \qquad \text{for } A, B, C \neq \tag{20}$$

and

$$\gamma_{ABB} = \beta_{B.A} = \mathrm{e}^{-\beta_A}\beta_{B|a}\delta^{a}_{A}.$$

Conformally-flat fields will be considered in § 7.

4. Type D fields

These fields are characterized by the condition

$$\alpha = \alpha_3 = -2\alpha_1 = -2\alpha_2.$$

If we choose the space-like triad so that equations (19) and (20) hold and specialize equation (17) to degenerate fields, we obtain, for $C = 1, 2$

$$\begin{aligned}&(\alpha-\tfrac{1}{3}\rho)_{,3}+3\alpha\beta_{C,3} = 0\\ &(\alpha+\tfrac{1}{6}\rho)_{,C}+\tfrac{3}{2}(\beta_{3,C}+U_{,C}) = 0\\ &(\alpha-\tfrac{1}{3}\rho)_{,C}+3\alpha U_{,C} = 0.\end{aligned} \tag{21}$$

We note that U cannot be constant for type D fields, since equation (8) implies that the 3-space is an Einstein space and hence of constant curvature (Eisenhart 1949). It follows from equation (14) that $\alpha_M = 0$, $M = 1, 2, 3$, and consequently the space is conformally flat.

On integrating the third equation of (21), we obtain

$$\alpha = \mathrm{e}^{-3U}\left(\tfrac{1}{3}\int \mathrm{e}^{3U}\rho'(u)\,\mathrm{d}u+\mathrm{e}^{3Z(z)}\right) \tag{22}$$

where $z = x^3$ and $Z(z)$ is an arbitrary function of integration. Eliminating $\rho_{,C}$ from the second and third of equations (21), we obtain

$$\beta_{3,C} = -\alpha^{-1}\alpha_{,C}-2U_{,C}$$

and hence

$$\mathrm{e}^{\beta_3} = \alpha^{-1}\,\mathrm{e}^{-2U}f(z). \tag{23}$$

From the first equation of (21) it follows that:

$$\beta_C = \beta(x^1, x^2, z)+\bar{\beta}_C(x^1, x^2) \qquad \beta_{|13} = -\tfrac{1}{3}\alpha^{-1}(\alpha-\tfrac{1}{3}\rho)_{|3}. \tag{24}$$

Thus the metric takes the form

$$G = \mathrm{e}^{2\beta}\,\mathrm{d}\sigma^2+f^2(z)\alpha^{-2}\,\mathrm{e}^{-4U}\,\mathrm{d}z^2-\mathrm{e}^{2U}\,\mathrm{d}t^2 \tag{25}$$

where $\mathrm{d}\sigma^2$ is a two dimensional metric form dependent only on x^1 and x^2.

The fields may be divided into four classes:

(i) ρ = constant
(ii) $\rho = \rho(z)$ that is $\rho_{|[\alpha}e_{3\beta]} = 0$, $\rho_{|\alpha}e_3^\alpha \neq 0$
(iii) $\rho = \rho(x^1, x^2)$ that is $\rho_{|[\alpha}e_{3\beta]} \neq 0$, $\rho_{|\alpha}e_3^\alpha = 0$
(iv) $\rho = \rho(x^1, x^2, z)$ that is $\rho_{|[\alpha}e_{3\beta]} \neq 0$, $\rho_{|\alpha}e_3^\alpha \neq 0$.

The classification is invariant since e_3^α is uniquely determined when u^α is, that is when $p+\rho \neq 0$. The condition $p+\rho = 0$ has already been shown to imply ρ = constant, and consequently all fields with $p+\rho = 0$ belong to class (i).

5. Space–times of class (i) (constant density)

Using the fact that ρ is constant in equations (22) and (24) and eliminating U from equation (23) by means of equation (22), we see that the metric takes the form

$$G = \alpha^{-2/3}(\mathrm{d}\sigma^2 + \mathrm{d}z^2 - \mathrm{e}^{2Z}\,\mathrm{d}t^2). \tag{26}$$

The function of z before the $\mathrm{d}z^2$ term in the metric has been eliminated by a coordinate transformation of the form $\tilde{z} = \tilde{z}(z)$. For the case of constant density, equation (11) may be integrated to obtain

$$p + \rho = 2A\,\mathrm{e}^{-U}. \tag{27}$$

We note that equation (26) is identical with equation (3.2.12) of JEK if we put $\alpha = \pm\mathrm{e}^{-3\tau}$. Hence we can follow JEK in subdividing metrics of the form (26) into the four classes: (*a*) the hypersurface, $\alpha =$ constant, contains the space-like eigenblade of the Weyl tensor, that is, $\alpha = \alpha(z)$; (*b*) the invariant α is constant; (*c*) the hypersurface, $\alpha =$ constant, contains the time-like eigenblade of the Weyl tensor, that is, $\alpha = \alpha(x^1, x^2)$; (*d*) the hypersurface, $\alpha =$ constant, contains neither eigenblade of the Weyl tensor, that is, $\alpha = \alpha(x^1, x^2, z)$.

5.1. Case (*a*)

In this case, by means of the coordinate transformation $z = \pm\alpha^{-1/3}$, the metric can be expressed in the form

$$G = z^2\,\mathrm{d}\sigma^2 + V^2(z)\,\mathrm{d}z^2 - \exp(2U(z))\,\mathrm{d}t^2. \tag{28}$$

On using (A.3) and (A.4) in the Appendix, with $W = z$, the field equations become

$$-\frac{K}{z^2} + V^{-2}\left(\frac{U_{|3}}{z} - \frac{V_{|3}}{zV} + 1\right) = \frac{1}{2}(p - \rho) \tag{29a}$$

$$V^{-2}\left(U_{|33} + U_{|3}^2 - \frac{V_{|3}U_{|3}}{V} - \frac{2V_{|3}}{zV}\right) = \frac{1}{2}(p - \rho) \tag{29b}$$

$$V^{-2}\left(U_{|33} + U_{|3}^2 - \frac{V_{|3}U_{|3}}{V} + \frac{2U_{|3}}{z}\right) = \frac{1}{2}(\rho + 3p) \tag{29c}$$

where $K = K(x^1, x^2)$ is the gaussian curvature of $\mathrm{d}\sigma^2$. Since K is the only function in equation (29*a*) that depends on x^1 and x^2 it must be constant. By a change in the scale of z we may set $K = \pm 1$, or 0, and choose coordinates θ and ϕ such that

$$\mathrm{d}\sigma^2 = \mathrm{d}\theta^2 + f^2(\theta)\,\mathrm{d}\phi^2 \tag{30}$$

where

$$f(\theta) = \sin\theta, \theta, \sinh\theta \qquad \text{for } K = +1, 0, -1$$

respectively.

From equations (29*a*) and (29*c*), we obtain

$$\frac{K}{z^2} - \frac{1}{V^2 z^2} + \frac{2V_{|3}}{zV^3} = \rho$$

which, when integrated, gives

$$V^{-2} = K - \tfrac{1}{3}\rho z^2 - 2mz^{-1} \tag{31}$$

where m is constant. A second consequence of equations (29a and c) is the equation

$$V^{-2}z^{-1}(U_{|3} + V^{-1}V_{|3}) = \tfrac{1}{2}(p+\rho).$$

On using equations (27) and (31) we obtain

$$(K - \tfrac{1}{3}\rho z^2 - 2mz^{-1})U_{|3}\,\mathrm{e}^U + (\tfrac{1}{3}\rho z - mz^{-2})\,\mathrm{e}^U = Az$$

and, on integration

$$\mathrm{e}^U = (K - \tfrac{1}{3}\rho z^2 - 2mz^{-1})\left(B + A\int z(K - \tfrac{1}{3}\rho z^2 - 2mz^{-1})^{-3/2}\,\mathrm{d}z\right). \tag{32}$$

In general, equation (32) involves elliptic integrals, but for $m = 0$ it becomes

$$\mathrm{e}^U = C + B(K - \tfrac{1}{3}\rho z^2)^{1/2} \tag{32a}$$

whereas for $p+\rho = 0$ (ie $A = 0$) equation (32) becomes

$$\mathrm{e}^U = V^{-1}. \tag{32b}$$

Using equation (A.3) we see that the invariant α is given by

$$\alpha = -2mz^{-3}.$$

If $m = 0$, it follows that $\alpha = 0$ and hence that the space is conformally flat.

The metric (28) admits a four-parameter isometry group $G_4 = G_3 \times G_1$, and G_3 which acts in the (x^1, x^2) surface contains a one dimensional isotropy subgroup. For $K = 1$, G_3 is isomorphic to the group of the sphere and the space is spherically symmetric. For $K = 0$, G_3 is isomorphic to the isometry group of the plane, whereas for $K = -1$, G_3 is isomorphic to the group of the two dimensional Lobatschewski space.

If $K = 1$ and $m = 0$ the solution is the interior solution of Schwarzschild (1916) and if $K = 1$ and $A = 0$ it is the exterior solution of Schwarzschild (1916) with a cosmological constant. These solutions generalize those obtained in JEK (case a) for vacuum fields to the case of a perfect fluid of constant density.

5.2. Case (b)

The metric (26) becomes in this case

$$G = \mathrm{d}\sigma^2 + \mathrm{d}z^2 - \exp(2Z(z))\,\mathrm{d}t^2.$$

Using equations (A.3) and (A.4) with $V = W = 1$, the field equations (8) become

$$K = -\tfrac{1}{2}(p-\rho)$$

and

$$Z_{|33} + Z_{|3}^2 = \tfrac{1}{2}(p-\rho) = \tfrac{1}{2}(\rho+3p)$$

where K is the gaussian curvature of $\mathrm{d}\sigma^2$. It follows from these equations that $p+\rho = 0$ and $K = \rho$. Further, from equation (A.2) we note that $-(Z_{|33} + Z_{|3}^2)$ is the gaussian curvature of $\mathrm{d}\sigma_2^2 = \mathrm{d}z^2 - \exp(2Z)\,\mathrm{d}t^2$, and so the space is the direct product of two

two dimensional spaces of the same constant curvature. In fact the metric is given explicitly by

$$G = d\theta^2 + \sin^2(\sqrt{\rho}\theta)\, d\phi^2 + dz^2 - \sin^2(\sqrt{\rho}z)\, dt^2 \qquad \text{for } \rho > 0$$

and

$$G = d\theta^2 + \sinh^2(\sqrt{(-\rho)}\theta)\, d\phi^2 + dz^2 - \sinh^2(\sqrt{(-\rho)}z)\, dt^2 \qquad \text{for } \rho < 0.$$

If $\rho = 0$ the space–time is flat. The invariant α is given by $\alpha = -\frac{2}{3}\rho$. The above solutions all possess a six-parameter group of (local) isometries. If $\rho \neq 0$, the space–time is of type D and so the group is complete (EK).

5.3. *Cases (c) and (d)*

It is convenient to consider these two cases together. We write $\alpha = \pm e^{-3\tau}$, and introduce a metric G'' by $G'' = G' \exp\{-(4\tau + 2Z)\}$, that is, $G'' = d\sigma^2 + dz^2$. On using equation (A.1) with $n = 3$ and $V = \exp\{-(2\tau + Z)\}$, we obtain

$$R'_{ab} = R''_{ab} - \exp(2\tau + Z)(\exp\{-(2\tau + Z)\})_{\|ab} + \exp\{-(2\tau + Z)\} \times (\exp(2\tau + Z))\|^{c}_{\|c} g''_{ab}$$

where covariant derivatives are taken with respect to G'' and R'_{ab} and R''_{ab} are the Ricci tensors of G' and G'' respectively. On substituting the above equation into equation (8)$'$, we obtain

$$R''_{ab} - 2\, e^{\tau}(e^{-\tau})_{\|ab} + e^{-Z}(e^{Z})_{\|ab} = [2p\, e^{2\tau} - \exp\{-(2\tau + Z)\} \times (\exp(2\tau + Z))\|^{c}_{\|c}] g''_{ab}. \qquad (33)$$

This equation differs from the equation (3.2.29) in JEK by the term involving p, and hence we may use the proof in JEK to show that, in appropriate coordinates (X, ϕ, z, t), the metric takes the form

$$G = (X + Y)^{-2}(f^{-1}\, dX^2 + f\, d\phi^2 + dz^2 - e^{2Z}\, dt^2) \qquad (34)$$

where $X + Y = e^{-\tau}$, $Y = Y(z)$, $Z = Z(z)$ and $f = f(X)$. In case (c), since $\tau_{|3} = 0$, it follows that Y is constant and, by a suitable translation of the coordinate X, may be put equal to zero. The metric is of the form

$$G = X^{-2}(f^{-1}\, dX^2 + f\, d\phi^2 + dz^2 - e^{2Z}\, dt^2). \qquad (35)$$

On using equations (8) and (9) we obtain

$$e^{-U}(e^{U})\|^{\lambda}_{\|\lambda} - \tilde{R}^3_3 = p + \rho$$

where $\lambda = 1, 2$. Using equation (A.5) with $V = X^{-1}$ and remembering that $e^U = X^{-1}\, e^Z$, we obtain $p + \rho = 0$.

It is unnecessary to integrate the remaining field equations, since by means of the complex coordinate transformation: $x = X^{-1}$, $\tilde{\phi} = it$, $\tilde{t} = i\phi$ the metric (35) and field equations may be put in a form identical with that of equations (28) and (29) with $p + \rho = 0$. With the aid of equations (31) and (32*b*) we obtain the following solution:

$$G = (K + 2mr^{-1} - \tfrac{1}{3}\rho r^2)^{-1}\, dr^2 + (K + 2mr^{-1} - \tfrac{1}{3}\rho r^2)\, d\phi^2 + r^2(d\theta^2 - f^2(\theta)\, dt^2)$$

where $f(\theta) = \sin\theta, \theta, \sinh\theta$ for $K = +1, 0, -1$ respectively. If G is to have the correct signature, $K + 2mr^{-1} - \frac{1}{3}\rho r^2 > 0$.

The invariant α is given by $\alpha = 2mr^{-3}$.

The solution has a complete four dimensional isometry group and a one dimensional isotropy group. The time-like eigensurface of the Weyl tensor (r = constant, ϕ = constant) has constant curvature K. This solution generalizes that of JEK (case b) to include vacuum solutions with a nonzero cosmological constant.

In case (d) $Y' \neq 0$ and $X' \neq 0$. Only two of the field equations (33) remain to be integrated. If we put $a = b = 3$ and use equation (A.5) with $V = 1$ we obtain

$$Z'' + Z'^2 - 2(X+Y)^{-1}Y'' = [2p\,e^{2\tau} - \exp\{-(2\tau+Z)\}(\exp(2\tau+Z))|^c_{\|c}]$$

and if we contract (33) we obtain

$$f'' + Z'' + Z'^2 + 2(X+Y)^{-1}(Y''+f') = 3[2p\,e^{2\tau} - \exp\{-(2\tau+Z)\}\times(\exp(2\tau+Z))|^c_{\|c}].$$

On using equation (A.6) with $g^{11} = (g^{22})^{-1} = f(X)$ and $g^{33} = 1$ we obtain

$$\exp\{-(2\tau+Z)\}(\exp(2\tau+Z))|^c_{\|c} = Z'' + Z'^2 - 2(X+Y)^{-1}(Y''+2Y'Z'+f') + 6(X+Y)^{-2}(Y'^2+f).$$

From the last three equations we obtain

$$(Z''+Z'^2)(X+Y)^2 - (2Y''+2Y'Z'+f')(X+Y) + 3(Y'^2+f) = p \tag{36a}$$

and

$$(Z''+Z'^2-\tfrac{1}{2}f'')(X+Y) - 2Y'' + f' = 0. \tag{36b}$$

In addition, equation (9)$'$ becomes, with the aid of equation (A.6)

$$(Z''+Z'^2)(X+Y)^2 - (Y''+3Y'Z'+f')(X+Y) + 3(Y'^2+f) = \tfrac{1}{2}(\rho+3p). \tag{37}$$

From equations (27), (36a) and (37) it can be seen that

$$Y'' - Y'Z' = A\,e^{-Z}. \tag{38}$$

Equation (36b) is equivalent to the equation

$$f''Y - 2(Z''+Z'^2)X = -f''X + 2f' - 4Y'' + 2(Z''+Z'^2)Y. \tag{39}$$

We note that the right hand side of this equation is the sum of a function of X, $h(X)$ say, and a function of z. On differentiating equation (39) twice with respect to X it follows that $f^{\mathrm{Iv}} = h'' = 0$, and hence

$$f = aX^3 + bX^2 + cX + d \qquad h = 2(bX+c) \tag{40}$$

where a, b, c and d are constants.

If we substitute the equations (40) in equation (39) we obtain

$$Z'' + Z'^2 - 3aY + b = 0 \qquad 2Y'' - (Z''+Z'^2)Y = c - bY.$$

Consequently

$$2Y'' = 3aY^2 - 2bY + c$$

and hence

$$Y'^2 = aY^3 - bY^2 + cY + \tilde{d} = g(Y) \tag{41}$$

where $\tilde{d}$ is a constant.

If a coordinate transformation is made so that z is replaced by Y, then the metric becomes

$$G = (X+Y)^{-2}(f^{-1}\,\mathrm{d}X^2 + f\,\mathrm{d}\phi^2 + g^{-1}\,\mathrm{d}Y^2 - \mathrm{e}^{2Z}\,\mathrm{d}t^2). \tag{42}$$

From equations (38) and (41) we obtain the equation

$$\frac{1}{2}g'(Y) - g(Y)\frac{\mathrm{d}Z}{\mathrm{d}Y} = A\,\mathrm{e}^{-Z}$$

which on integration gives

$$\mathrm{e}^Z = g^{1/2}\left(B + A\int (g(s))^{-3/2}\,\mathrm{d}s\right). \tag{43}$$

On substituting equation (43) in equation (36), we obtain, with the aid of equation (38)

$$3(d+\tilde{d}) + \rho = 0$$

and hence

$$f(X) = -g(-X) - \tfrac{1}{3}\rho. \tag{44}$$

The invariant α is given by

$$\alpha = a(X+Y)^3$$

and so if $a = 0$ the space is conformally flat. We shall assume that $a \neq 0$, and then by affine transformations of the coordinates X and Y and a change of scale of the coordinates ϕ and t we may set $a = \pm 1$ and $b = 0$ whilst preserving the form of the metric (42). The metric is given by equation (42), where $f = \pm X^3 + cX + d$ and g and Z are given by equations (43) and (44). There exist two functionally independent metric invariants

$$\alpha = \pm(X+Y)^3 \qquad \gamma^{-2}\gamma_{|i}\gamma^{|i} = \pm(X^3+Y^3) + c(X+Y) - \tfrac{2}{3}\rho$$

where $\gamma = \alpha^{1/3}$. Thus the trajectories of the isometry group are the two dimensional surfaces $X =$ constant, $Y =$ constant. Since the metric is type D, the isotropy group acts in either the (X, ϕ) surface or the (Y, t) surface and consequently must be discrete. Thus, the complete isometry group is two dimensional. From equations (27) and (43), we see that the pressure is given by

$$p = -\rho + 2Ag^{-1/2}(X+Y)\left(B + A\int g^{-3/2}\,\mathrm{d}s\right)^{-1}. \tag{45}$$

The above solution generalizes the solution C in JEK to the case of a uniform density perfect fluid and contains the solution in JEK as a special case.

6. Space–times of classes (ii–iv) (variable density)

In case (ii) $\rho = \rho(z)$, and from equation (12) it follows that $U = U(z)$ and hence from equations (22)–(24) we may deduce that

$$\alpha = \alpha(z) \qquad \beta_3 = \beta_3(z) \qquad \beta_C = \beta(z) + \bar{\beta}_C(x^1, x^2).$$

By means of the coordinate transformation $\tilde{z} = \exp(\beta(z))$ it follows that the metric (25) takes the form:

$$G = z^2\,\mathrm{d}\sigma^2 + V^2(z)\,\mathrm{d}z^2 - \exp(2U(z))\,\mathrm{d}t^2. \tag{46}$$

This equation is identical with equation (28) and we may use the analysis following equation (28) to show that $d\sigma^2$ is of constant curvature K, and consequently takes the form (30) and that

$$V^{-2} = K - z^{-1}\int_0^z \rho(s)s^2\,ds + 2mz^{-1}. \tag{47}$$

For this metric the invariant α is given by

$$\alpha = 2mz^{-3} + \tfrac{1}{3}z^{-3}\int_0^z \rho'(s)s^3\,ds.$$

The metric (46) admits a four-parameter isometry group with a one dimensional isotropy subgroup. For $K = 1$ the space is spherically symmetrical. Spherically symmetrical fluids with nonconstant density have been considered by a number of authors (Tolman 1939, Wyman 1949 and Bondi 1964) and many of their results have trivial generalizations to the other two cases ($K = 0, K = -1$) obtained above.

We shall consider classes (iii) and (iv) together. For these two classes $\rho_{|C} \neq 0$ for $C = 1, 2$ and in class (iii) the condition $\rho_{|3} = 0$ is also satisfied. The metric is of the form (25)

$$G = e^{-2U}(e^{2\gamma}\,d\sigma^2 + \alpha^{-2}\,e^{-2U}\,dz^2) - e^{2U}\,dt^2 \tag{25$'$}$$

where $\gamma = \beta + U$ and

$$\gamma_{|3} = U_{|3} - \tfrac{1}{3}\alpha^{-1}(\alpha - \tfrac{1}{3}\rho)_{|3}. \tag{24$'$}$$

With the aid of equation (A.7), the field equations (8)′ become when $a = 3$ and $b = 1, 2$

$$\gamma_{|3b} + \gamma_{|3}(\alpha^{-1}\alpha_{|b} + U_{|b}) + 2U_{|3}U_{|b} = 0.$$

Eliminating γ from this equation with the aid of equations (21) and (24)′ we obtain

$$2(U_{|3b} + U_{|3}U_{|b}) + \tfrac{2}{3}\alpha^{-1}(\alpha_{|3}U_{|b} + U_{|3}\alpha_{|b}) + \tfrac{1}{9}\alpha^{-1}(U_{|b}\rho_{|3} + U_{|3}\rho_{|b}) = 0.$$

With the aid of equations (12) and (21) we see that

$$U_{|3b} + 2U_{|3}U_{|b} + \tfrac{1}{3}\alpha^{-1}(\alpha_{|3}U_{|b} + 2\alpha_{|b}U_{|3}) = 0. \tag{48}$$

One consequence of equations (12), (21), and (48) is that

$$U_{|1}U_{|32} = U_{|2}U_{|31}$$

and hence $U = U(f(x^1, x^2), z)$ and, by means of the coordinate transformation $\tilde{x}^1 = X = f(x^1, x^2)$, we see that $U = U(X, z)$.

Equations (21) and (24)′ imply that

$$\alpha = \alpha(X, z) \qquad \gamma = \gamma(X, z).$$

On choosing $x^2 = \phi$, so that the x^1 and x^2 lines are orthogonal, we see that the metric has the form

$$G = e^{-2U}\{e^{2\gamma}(A^2(X, \phi)\,dX^2 + B^2(X, \phi)\,d\phi^2) + \alpha^{-2}\,e^{-2U}\,dz^2\} - e^{2U}\,dt^2.$$

The remaining field equations (8)′, on using equation (A.7) with $d^a_b = -\alpha\,e^U\gamma_{|3}\delta^a_b$, become

$$\alpha\,e^U(\alpha^{-1}\,e^{-U})_{\|a}^{|a} + 2(\gamma_{|33} + \gamma_{|3}^2 + \gamma_{|3}U_{|3} + U_{|3}^2 + \alpha^{-1}\alpha_{|3}\gamma_{|3})\alpha^2\,e^{2U} = 2p\,e^{-2U} \tag{49}$$

and

$$\tilde{R}^a_b + \alpha\, e^U(\alpha^{-1}\, e^{-U})\|^a_{\|b} + \alpha^2\, e^{2U}\delta^a_b(\gamma_{|33} + 2\gamma^2_{|3} + \gamma_{|3}U_{|3} + \alpha^{-1}\alpha_{|3}\gamma_{|3})$$
$$+ 2U^{|a}U_{|b} = 2p\, e^{-2U}\delta^a_b \tag{50}$$

where $\tilde{R}^a_b$ is the Ricci tensor of $H = e^{2\gamma}(A^2\, dX^2 + B^2\, d\phi^2)$ and $a, b = 1, 2$. For $a = 1$ and $b = 2$ equation (50) is $(\alpha^{-1}\, e^{-U})^{|1}_{|2} = 0$, that is, $A_{|2}(\alpha^{-1}\, e^{-U})_{|1} = 0$. Consequently, either $A = A(X)$ or $\alpha\, e^U = f^n(z)$. In the latter case equation (50) implies $U^{|a}U_{|b}$ is proportional to δ^a_b, and since $U_{|2} = 0$ it follows that $U = U(z)$ and, from equation (12), that $\rho = \rho(z)$. This contradicts the defining equations of classes (iii) and (iv). Hence it follows that $A = A(X)$. Subtracting the equation (50) with $a = b = 2$ from that with $a = b = 1$, we see that

$$\alpha\, e^U((\alpha^{-1}\, e^{-U})\|^1_{\|1} - (\alpha^{-1}\, e^{-U})\|^2_{\|2}) + 2U^{|1}U_{|1} = 0$$

which implies that $B^{-1}B_{|1} = f^n(X, z)$. Since $B = B(X, \phi)$, it follows that $B^{-1}B_{|1} = f^n(X)$ and consequently $B = \bar{B}(X)g(\phi)$. By means of coordinate transformations of the form $\tilde{X} = \tilde{X}(X)$, and $\tilde{\phi} = \tilde{\phi}(\phi)$, the metric may be put in the form

$$G = e^{-2U}\{e^{2\gamma}A^2(X)(dX^2 + d\phi^2) - \alpha^{-2}\, e^{-2U}\, dz^2\} - e^{2U}\, dt^2 \tag{51}$$

where $U = U(X, z)$, $\gamma = \gamma(X, z)$ and $\alpha = \alpha(X, z)$.

In class (iii) we have in addition $U_{|3} = 0$ and so equations (24)$'$ and (48) imply that $\alpha_{|3} = \gamma_{|3} = 0$. The metric is of the form (51) but in this case $U = U(X)$, $\gamma = \gamma(X)$ and $\alpha = \alpha(X)$. The metric (51) admits a two-parameter abelian isometry group in class (iv) and a three-parameter abelian group in class (iii). The metric can be put in the Weyl normal form

$$G = e^{-2U}\{e^{2\eta}(dX^2 + dz^2) + e^{2\delta}\, d\phi^2\} - e^{2U}\, dt^2$$

where $U = U(X, z)$, $\eta = \eta(X, z)$ and $\delta = \delta(X, z)$, by means of coordinate transformations of the form $\tilde{X} = \tilde{X}(X, z)$, $\tilde{z} = \tilde{z}(X, z)$. However, it seems easier to integrate the remaining field equations if the metric is left in the form (51).

If we write $e^\beta = A\exp(\gamma - U)$ and $V = \alpha^{-1}\, e^{-2U}$ in equation (51) and use equation (A.7) with $d^\mu_\nu = -\beta_{|3}V^{-1}\delta^\mu_\nu$ for $\mu, \nu = 1, 2$, we obtain

$$\tilde{R}^\mu_\nu = e^{-2\beta}(\beta_{|11} + V^{-1}\beta_{|1}V_{|1})\delta^\mu_\nu + V^{-2}(\beta_{|33} - V^{-1}\beta_{|3}V_{|3} + 2\beta^2_{|3})\delta^\mu_\nu$$
$$+ V^{-1}\, e^{-2\beta}(V_{|11} - 2\beta_{|1}V_{|1})\delta^\mu_1\delta^1_\nu$$
$$\tilde{R}^3_\mu = V^{-1}(V^{-1}\beta_{|3})_{|1}\delta^1_\mu \tag{52}$$

and

$$\tilde{R}^3_3 = V^{-1}\, e^{-2\beta}V_{|11} + 2V^{-2}(\beta_{|33} + \beta^2_{|3} - V^{-1}\beta_{|3}V_{|3}).$$

Since the metric is of type D and since the z direction is the distinguished Ricci eigenvector it follows that $\tilde{R}^3_\mu = 0$. Noting that $\tilde{R}^1_2 = 0$, we see that the X and ϕ directions are also Ricci eigenvectors and consequently $\tilde{R}^1_1 = \tilde{R}^2_2$. Using equation (52) we see that

$$V^{-1}\beta_{|3} = h(z) \qquad V_{|1}\, e^{-2\beta} = g(z). \tag{53}$$

It follows from the field equations (8) that

$$(e^U)\|^1_{\|1} = (e^U)\|^2_{\|2} \qquad (e^U)\|^3_{\|1} = 0$$

where the covariant derivatives are taken with respect to $\tilde{g}_{ab}$. These equations imply that

$$U_{|1}\exp(U-2\beta) = f(z) \qquad (V^{-1}U_{|3}\,\mathrm{e}^{U})_{|1} = V^{-1}\beta_{|3}U_{|1}\,\mathrm{e}^{U}. \tag{54}$$

We note that $f \neq 0$ since $U_{|1} \neq 0$ for classes (iii) and (iv). Equations (53) and (54) imply that

$$V = \frac{g}{f}\,\mathrm{e}^{U} + k(z) \qquad \frac{U_{|3}\,\mathrm{e}^{U}}{V} = h(z)\,\mathrm{e}^{U} + l(z). \tag{55}$$

If we eliminate α from equation (48) we obtain the equation

$$U_{|13} - \tfrac{1}{3}V^{-1}(U_{|1}V_{|3} + 2U_{|3}V_{|1}) = 0.$$

On substituting equation (55) in this equation and writing $m(z) = g/f$, we obtain the equation

$$U_{|1}\{3lk^2\,\mathrm{e}^{-U} + k' + (3lm^2 + m')\,\mathrm{e}^{U}\} = 0.$$

Since $U_{|1} \neq 0$, it follows that either

$$l = k' = m' = 0 \qquad \text{or} \qquad k = l + \tfrac{1}{3}m^{-2}m' = 0.$$

If $k = 0$, it follows from equations (22), (55) and the definition of V that

$$m^{-1} = \tfrac{1}{3}\int \rho'(U)\,\mathrm{e}^{3U}\,\mathrm{d}U + \exp(3Z(z)).$$

Consequently $\int \rho'(U)\,\mathrm{e}^{3U}\,\mathrm{d}U$ is a function of z only and hence either $U = U(z)$ or $\rho =$ constant. In either case this contradicts the equation defining classes (iii) and (iv). It follows that the equations $l = k' = m' = 0$ must be valid. On substituting these equations in (54), (55) we see that

$$V = m\,\mathrm{e}^{U} + k \qquad \beta_{|3} = U_{|3} = h(z)V.$$

On integrating these equations, we find that

$$\mathrm{e}^{2\beta} = H(X)\,\mathrm{e}^{2U} \qquad \mathrm{e}^{U} = k(G(X)n(z) - m)^{-1}$$

where $H(X)$ and $G(X)$ are arbitrary functions of integration and $n(z) = \exp(\int kh\,\mathrm{d}z)$. From the first of the equations (54) it follows that

$$H(X)f(z) = -kG'(X)n(z)$$

and hence $H(X)$ is proportional to $G'(X)$. If we make the coordinate transformation $\tilde{X} = -1/G(X)$ and then make suitable scale changes in the other coordinates, we may write the metric in the form

$$G = (n + mX)^{-2}(F^{-1}\,\mathrm{d}X^2 + F\,\mathrm{d}\phi^2 + \mathrm{d}z^2 - X^2\,\mathrm{d}t^2) \tag{56}$$

where $F = F(X)$. The field equations (8) and (9) that are not identically satisfied are

$$\frac{1}{2}(n+mX)F'' - \frac{1}{2}(n+mX)\left(3m - \frac{n}{x}\right)F' + m\left(2m - \frac{n}{x}\right)F - (n+mX)n'' + 3n'^2 = \frac{1}{2}(p-\rho) \tag{57a}$$

$$-(n+mX)mF' + \left(2m - \frac{n}{x}\right)mF - 3(n+mX)n'' + 3n'^2 = \frac{1}{2}(p-\rho) \tag{57b}$$

and

$$\left(m+\frac{n}{x}\right)nF' - 3mnX^{-1}F - (n+mX)n'' + 3n'^2 = \frac{1}{2}(\rho+3p). \tag{57c}$$

On subtracting (57*b*) from (57*a*) we obtain

$$4n'' + n\left(F'' + \frac{F'}{X}\right) + mX\left(F'' - \frac{F'}{X}\right) = 0. \tag{58}$$

It follows, on differentiating equation (58) with respect to z, that either $n'(z) = 0$ or $F'' + F'/X = 4a$, where a is a constant.

In the first case the metric (56) is independent of z and so these cases belong to class (iii). The remaining solutions will be shown to belong to class (iv) or to be conformally flat. In the case where $n' = 0$ equation (58) becomes

$$F''(n+mX) + \frac{(n-mX)F'}{X} = 0.$$

On integrating we find

$$F = b + c(n^2 \ln X + 2mnX + \tfrac{1}{2}m^2X^2) \tag{59}$$

where b and c are constants. From the equations (57) we obtain the following expressions for the pressure and density:

$$2p = cX^{-2}(n+mX)^3(n-mX) + 2m\left(m - \frac{2n}{X}\right)F$$

$$2\rho = cX^{-2}(n+mX)^3(n+3mX) - 6m^2F.$$

The invariant α is given by

$$\alpha = -\tfrac{1}{3}ncX^{-2}(n+mX)^3.$$

Clearly the space is conformally flat if $nc = 0$. If $nc \neq 0$ the metric admits a complete three dimensional abelian local isometry group. The completeness follows from the fact that trajectories of the complete group must be contained in the hypersurface X = constant and so the group is of dimension at most six. However, the isotropy group is of dimension at most one since the metric is of type D and acts in the surface X = constant, ϕ = constant. Since the vector u^α is unique it follows that the isotropy group is discrete and that the complete isometry group is of dimension three.

If however $n' \neq 0$, $F'' + F'/X = 4a$. On integration we obtain the equation

$$F = aX^2 + b\ln X + c \tag{60}$$

where b and c are constants. From equation (58) it follows that

$$4(n'' + an) + d = 0 \qquad m(XF'' - F') = 0 \tag{61}$$

where d is constant. Using equations (60) and (61) we see that $d = 0$ and $mb = 0$. On integrating the first of equations (61) we obtain $n = Af(z)$, where $f(z) = \sin(\sqrt{a}z + B)$, $z+B$, $\sinh(\sqrt{(-a)}z + B)$ for $a > 0$, $a = 0$, $a < 0$ respectively.

On using the equation (A.7) we see that the invariant α is given by

$$\alpha = -\tfrac{1}{3}bn^2X^{-2}.$$

If the space–time is not conformally flat, $b \neq 0$ and consequently $m = 0$. Thus the metric (56) has the form

$$G = n^{-2}(F^{-1}\,\mathrm{d}X^2 + F\,\mathrm{d}\phi^2 + \mathrm{d}z^2 - X^2\,\mathrm{d}t^2)$$

where $F = F(X)$ is given by equation (60). On using equations (57) we obtain the following expressions for the pressure and energy-density of the matter:

$$\begin{aligned} 2p &= 6A^2I(a) + bn^2X^{-2} \\ 2\rho &= -6A^2I(a) + bn^2X^{-2} \end{aligned} \tag{62}$$

where $I(a) = |a|$, 1 for $a \neq 0$, $a = 0$ respectively.

A second (functionally independent of α) invariant is given by

$$\gamma = \tfrac{1}{4}\alpha^{-2}\alpha_{|i}\alpha^{|i} = (b \ln X + c)n^2X^{-2} + A^2I(a).$$

Thus we may use arguments similar to those for class (id) to show that the complete local isometry group is an abelian two dimensional group. It is easily seen from equations (62) that the equation of state of the matter is $p = \rho + 6A^2I(a)$.

7. Conformally flat space-times

In this case $C_{\alpha\beta\gamma\delta} = 0$ and so from equation (13) it follows that $\alpha_M = 0$ for $M = 1, 2, 3$. From equation (14) we see that $P_{AC} = 0$ and consequently the space cross sections t = constant, are Einstein spaces of dimension three. Hence the space is of constant curvature $K = \frac{1}{3}\rho$ (Eisenhart 1949). Consequently we may choose coordinates such that the metric takes the form

$$G = (1 - Kr^2)^{-1}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) - V^2(r, \theta, \phi)\,\mathrm{d}t^2. \tag{63}$$

The field equations (8) and (9) become

$$V^{-1}V|^{a}_{\|b} = \tfrac{1}{6}(\rho + 3p)\delta^a_b$$

and (64)

$$V^{-1}V|^{a}_{\|a} = \tfrac{1}{2}(\rho + 3p).$$

The integration of equations (64) is a long but straightforward calculation and eventually one obtains

$$\begin{aligned} V &= (B \sin\phi + C\cos\phi)r\sin\theta + Dr\cos\theta \\ &\quad + E(1 - Kr^2)^{1/2} + A/K \qquad \text{for } K \neq 0 \\ V &= (B\sin\phi + C\cos\phi)r\sin\theta + Dr\cos\theta + E + \tfrac{1}{2}Ar^2 \qquad \text{for } K = 0 \end{aligned} \tag{65}$$

where A, B, C, D, E are all constants, A being given by equation (27). By means of a rotation we set $B = C = 0$ in equation (65) whilst preserving the form of the metric (63). This solution of the field equations was obtained in a slightly different coordinate system by Stepanyuk (1968). It will be shown below that the solution in fact belongs to class (ia) with $m = 0$.

In the following it will be convenient to use cartesian coordinates defined by

$$\begin{aligned} x &= x^1 = r\sin\theta\sin\phi \qquad y = x^2 = r\sin\theta\cos\phi \\ z &= x^3 = r\cos\theta. \end{aligned} \tag{66}$$

If $K = 0$, equation (63) with equations (65), (66) becomes

$$G = \mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2-(Dz+\tfrac{1}{2}Ar^2+E)^2\,\mathrm{d}t^2. \tag{67}$$

By means of the coordinate transformation $\tilde{z} = z+D/A$, the metric may be expressed as

$$G = \mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2-(\tfrac{1}{2}Ar^2+E-D^2/A^2)^2\,\mathrm{d}t^2.$$

This metric is clearly spherically symmetrical. Since $\rho = 3K = 0$ it follows from equation (27) that:

$$p = \frac{2A}{\frac{1}{2}Ar^2+E-D^2/A^2}.$$

The above analysis is not valid if $A = 0$ but in this case the metric (67) is flat. The transformation to Minkowski coordinates being given by $\tilde{z} = (z+E/D)\cosh Dt$, $\tilde{t} = (z+E/D)\sinh Dt$.

If $K \neq 0$ it is convenient to rewrite the metric (63) in isotropic coordinates (R, θ, ϕ, t) where R is defined by $r = R(1+\frac{1}{4}KR^2)^{-1}$. We obtain

$$G = \left(1+\frac{1}{4}KR^2\right)^{-2}\left\{\mathrm{d}R^2+R^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\right.$$
$$\left.-\left(\frac{A}{K}+E\left(1-\frac{1}{4}KR^2\right)+Dr\cos\theta\right)^2\mathrm{d}t^2\right\}. \tag{68}$$

The space sections being of constant curvature admit six independent Killing vector fields given by (Robertson and Noonan 1968)

$$\xi^a_A = (1-\tfrac{1}{4}KR^2)\delta^a_A+\tfrac{1}{2}Kx^a\delta_{Ab}x^b \qquad \zeta^a_A = -\epsilon_{ABC}\delta^a_B\delta_{Cb}x^b. \tag{69}$$

It is easily verified that the three 4-vectors η^a_A defined by

$$\eta^a_1 = D\xi^a_1+EK\zeta^a_2 \qquad \eta^a_2 = D\xi^a_2-EK\zeta^a_1 \qquad \eta^a_3 = \zeta^a_3$$
$$\eta^4_A = 0 \tag{70}$$

are Killing vectors of the space–time. Since these vectors lie in the two dimensional surface $V =$ constant, $t =$ constant, this surface is of constant curvature and using the analysis of § 5, we see that the metric takes the form

$$G = (\alpha-Kr^2)^{-1}\,\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+f^2(\theta)\,\mathrm{d}\phi^2)-\left(\frac{A}{K}+E_1(\alpha-Kr^2)^{1/2}\right)\mathrm{d}t^2 \tag{71}$$

where $f(\theta) = \sin\theta, \theta, \sinh\theta$, for $\alpha = +1, 0, -1$, respectively.

The commutation relations of the operators $X_A(X_A = \eta^\alpha_A\,\partial/\partial x^\alpha)$ are

$$[X_1, X_2] = -(D^2K+E^2K^2)X_3 \qquad [X_2, X_3] = -X_1$$
$$[X_3, X_1] = -X_2. \tag{72}$$

Hence if $D^2K+E^2K^2 > 0, = 0, < 0$ the group is isomorphic to that of the two dimensional sphere, plane, Lobatschewski space, respectively and in the metric (71) $\alpha = +1, 0, -1$ respectively.

If $\rho = 3K > 0$, we note that $D^2K+E^2K^2 > 0$ unless $D = E = 0$ in which case the metric (63) is that of the Einstein universe.

Table 1

Class	Metric	α	Pressure and density	r	s
ia	$G = r^2(\mathrm{d}\theta^2 + f^2(\theta)\,\mathrm{d}\phi^2) + (F(r))^{-1}\,\mathrm{d}r^2 - F(r)\left(B + A\int rF^{-3/2}\,\mathrm{d}r\right)^2 \mathrm{d}t^2$ $F(r) = \alpha - \frac{1}{3}\rho r^2 - 2m/r, f(\theta) = \sin\theta, \theta, \sinh\theta$ for $\alpha = +1, 0, -1$	$-\frac{2m}{r^3}$	$p = -\rho + 2AF^{-1/2}\left(B + A\int rF^{-3/2}\,\mathrm{d}r\right)^{-1}$	4	1S
ib	$G = \mathrm{d}\theta^2 + h^2(\theta)\,\mathrm{d}\phi^2 + \mathrm{d}z^2 - h^2(z)\,\mathrm{d}t^2$ $h(\theta) = \sin(\sqrt{\rho}\theta), \rho > 0, h(\theta) = \sinh(\sqrt{(-\rho)}\theta)$ for $\rho < 0$	$-\frac{2}{3}\rho$	$p = -\rho$	6	2S, T
ic	$G = (F(r))^{-1}\,\mathrm{d}r^2 + F(r)\,\mathrm{d}\phi^2 + r^2(\mathrm{d}\theta^2 - f^2(\theta)\,\mathrm{d}t^2)$ F and f as in ia	$-\frac{2m}{r^3}$	$p = -\rho$	4	1T
id	$G = (X+Y)^{-2}\{(f(X))^{-1}\,\mathrm{d}X^2 + f(X)\,\mathrm{d}\phi^2 + (g(Y))^{-1}\,\mathrm{d}Y^2 + \mathrm{e}^{2Z}\,\mathrm{d}t^2\}$ $f(X) = \pm X^3 + cX + d, g(X) = -f(-X) - \frac{1}{3}\rho, \mathrm{e}^Z = g^{1/2}(Y)\left(B + A\int g^{-3/2}\,\mathrm{d}Y\right)$	$\pm(X+Y)^3$	$p = -\rho + 2A(X+Y)\mathrm{e}^{-Z}$	2	0
ii	$G = r^2(\mathrm{d}\theta^2 + f^2(\theta)\,\mathrm{d}\phi^2) + \left(\alpha - r^{-1}\int \rho(r)r^2\,\mathrm{d}r - 2mr^{-1}\right)^{-1}\mathrm{d}r^2 - \exp(2U(r))\,\mathrm{d}t^2$ $f(\theta)$ as in class (ia) U given by equations (29)	$\frac{1}{3}r^{-3}\int \rho'(r)r^3\,\mathrm{d}r$ $-2mr^{-3}$	$p = p(r)$ $\rho = \rho(r)$	4	1S
iii	$G = (n+X)^{-2}\{(F(X))^{-1}\,\mathrm{d}X^2 + F(X)\,\mathrm{d}\phi^2 + \mathrm{d}z^2 - X^2\,\mathrm{d}t^2\}$ $F(X) = b + c(n^2\ln X + 2nX + \frac{1}{2}X^2)$	$-\frac{n}{3}cX^{-2}(n+X)^3$	$2p = cX^{-2}(n+X)^3(n-X) - 2(n-2/X)F(X)$ $2\rho = cX^{-2}(n+X)^3(n+3X) - 6F(X)$	3	0
iv	$G = (n(z))^{-2}\{(F(X))^{-1}\,\mathrm{d}X^2 + F(X)\,\mathrm{d}\phi^2 + \mathrm{d}z^2 - X^2\,\mathrm{d}t^2\}$ $F = aX^2 + b\ln X + c$ $n(z) = A\sin(\sqrt{a}z + B), Az + B, A\sinh(\sqrt{(-a)}z + B)$ for $a > 0, = 0, < 0$	$-\frac{1}{3}bn^2(z)X^{-2}$	$2p = 6A^2I(a) - bn^2(z)X^{-2}$ $2\rho = -6A^2I(a) - bn^2(z)X^{-2}$ $I(a) = \lvert a\rvert$ for $a \neq 0$, $I(a) = 1, a = 0$	2	0

Consequently we have proved that the only positive density conformally flat static perfect fluid solutions are locally isomorphic to the interior Schwartzschild solution or one of the Einstein or de Sitter universes.

8. Conclusions

All degenerate static perfect fluid solutions of Einstein's equations have been found. These are all found to possess at least a two-parameter group of local isometries. The solutions obtained are displayed in table 1. The dimensions of the complete isometry and isotropy groups are given in columns 5 and 6 respectively. In column 6 the letter S or T denotes that the isotropy group acts in the space-like or time-like eigensurface of the Weyl tensor respectively.

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Appendix

For the reader's convenience we list below a number of useful formulae taken from JEK.

If in an n dimensional pseudoriemannian manifold there exist two conformally related metrics G and $\bar{G}$ where

$$G = V^2\bar{G}$$

then

$$\bar{R}_{ik} = R_{ik} - (n-2)V^{-1}V_{\|ik} + (n-2)^{-1}V^{n-2}(V^{2-n})_{\|}{}^{l}{}_{\|l}\,g_{ik} \tag{A.1}$$

where covariant derivatives are taken with respect to G.

For a two dimensional metric of the form

$$\mathrm{d}\sigma^2 = (W\,\mathrm{d}x^1)^2 \pm (V\,\mathrm{d}x^2)^2$$

the following equations are valid:

$$R^i_k = -K\delta^i_k \qquad R^{ij}_{kl} = -2K\delta^i_{[k}\delta^j_{l]}$$

where

$$K = -\frac{1}{VW}\left(\left(\frac{V_{|1}}{W}\right)_{|1} \pm \left(\frac{W_{|2}}{V}\right)_{|2}\right). \tag{A.2}$$

For a three dimensional metric of the form

$$G = W^2(x^3)\,\mathrm{d}\sigma^2 + (V(x^3)\,\mathrm{d}x^3)^2$$

where $\mathrm{d}\sigma^2$ is a two dimensional metric involving x^1 and x^2 only, we can choose x^1 and

x^2 so that the coordinate lines are Ricci eigendirections. Further the equations

$$R^1_1 = R^2_2 = \frac{1}{W^2}\left(-K+\frac{1}{V}\left(\frac{WW_{|3}}{V}\right)_{|3}\right) \qquad R^3_3 = \frac{2}{VW}\left(\frac{W_{|3}}{V}\right)_{|3}$$

$$R^\mu_v = 0 \qquad \text{for } \mu \neq v \tag{A.3}$$

where K is the gaussian curvature of $d\sigma^2$, are valid.

For this metric

$$\Gamma^i_{3j} = \frac{W_{|3}}{W}\delta^i_j \qquad \Gamma^3_{i3} = \Gamma^i_{33} = 0 \qquad \Gamma^3_{33} = \frac{V_{|3}}{V} \qquad \text{for } i,j = 1,2$$

and if $F = F(x^3)$ then

$$F|^i_{\|j} = \frac{W_{|3}}{V^2 W}F'\delta^i_j \qquad F|^3_{\|i} = 0 \qquad F|^3_{\|3} = V^{-2}(F''-V^{-1}F'V_{|3}). \tag{A.4}$$

For a metric of the form

$$G = d\sigma^2+(V(x^i)\,dx^3)^2 \qquad i,j = 1,2$$

the Ricci tensor is given by

$$R^i_j = V^{-1}V|^i_{\|j}-K\delta^i_j \qquad R^i_3 = 0 \qquad R^3_3 = V^{-1}V|^i_{\|i}. \tag{A.5}$$

For this metric we note that, since $\Gamma^a_{3b} = \Gamma^3_{ab} = 0$ for $a, b = 1, 2, 3$, $U_{\|ij} = U_{;ij}$, where a semicolon denotes a covariant derivative made with $d\sigma^2$.

For a metric of the form $G = g_{\mu v}\,dx^\mu\,dx^v$ where $\mu, v = 1, 2, \ldots, n$, and $\det(g_{\mu v}) = 1$, it follows that $V|^\mu_{\|\mu} = (V_{|\mu}g^{\mu\lambda})_{|\lambda}$, and if $g_{\mu v} = 0$ for $\mu \neq v$

$$V|^\mu_{\|\mu} = \sum_{\mu=1}^{n} (V_{|\mu}g^{\mu\mu})_{|\mu}. \tag{A.6}$$

Finally for a metric of the form

$$G = g_{\mu v}(x^k)\,dx^\mu\,dx^v+(V(x^k)\,dx^n)^2$$

where $\mu, v = 1, 2, \ldots, n-1$ and $k = 1, 2, \ldots, n$, the second fundamental form $D = d_{\mu v}\,dx^\mu\,dx^v$ of the hypersurfaces x^n = constant is given by $d_{\mu v} = -\frac{1}{2}V^{-1}\partial(g_{\mu v})/\partial x^n$. The Ricci tensor of this metric is given by

$$R^\mu_v = \bar{R}^\mu_v + V^{-1}(V|^\mu_{\|v} - \dot{d}^\mu_v)+dd^\mu_v$$

$$R^n_\mu = V^{-1}(d^v_\mu - d\delta^v_\mu)_{\|v} \tag{A.7}$$

and

$$R^n_n = V^{-1}(V|^v_{\|v}-\dot{d})+d_{\mu v}d^{\mu v}$$

where a dot signifies $\partial/\partial x^n$, $\bar{R}^\mu_v$ is the Ricci tensor of $g_{\mu v}$ and where covariant derivatives are taken with respect to $g_{\mu v}$.

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